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Stability and Coefficients Properites of Polynomials of Linear Discrete Systems

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In this paper, by examining the coefficients of a given polynomial, we derive sufficient conditions for the zeros of the polynomial to be either inside the unit disk in the complex plane or at least one zero not inside the unit disk. The results are easily verifiable and provide handy ways of checking. Most results are proved using either Rouché's Theorem or fundamental mathematical ideas. Some of the results are extensions of known coefficient properties found in the literature.

Key Words: Discrete Systems, Linear Systems, Polynomials, Stability

1. Introduction

Understanding the stability for a given system with least amount of efforts but in a precise manner has long been one of the main interests for scientists and engineers. The transfer function of a one-dimensional discrete system is bounded-input bounded-output stable, if there are no poles in or on the boundary of the unit circle in the complex plane. Therefore properties of the coefficients of a polynomial $P(z)$ which guarantees the location of the zeros of $P(z)$ with respect to the unit circle are important in the analysis and design of discrete systems. Many conditions on the coefficients of $P(z)$ are known, see for examples, (Marden, 1949; Jury, 1964; Heinen, 1985; Ramachandran and Ahmadi, 1987). However, there are continuing interests for additional properties and simplified conditions and proofs of known properties. Recently several stability results have appeared in the literature including continuous systems (Bauer and Jury, 1991; Benidir and Picinbono, 1991; Bentsman and Hong, 1991; Bentsman et al., 1991; Hong and Bentsman, 1992a, b; Hong and Wu, 1992, Wu and Hong, 1992) to name a few.

In this paper, we establish sufficient conditions on the coefficients of a polynomial $P(z)$ such that either all the zeros of $P(z)$ are inside the unit circle, or at least one zero of $P(z)$ is not inside the unit circle. Some of the conditions presented are extensions of known coefficient properties. Also some examples are included to illustrate the conditions.

The results established here are based on elementary mathematical ideas and Rouché's Theorem (Lang, 1985, p. 172) which is stated below for completeness.

Theorem(Rouché) :If $f(z)$ and $g(z)$ are analytic inside and on a closed contour C in the complex plane and $|f(z)|>|g(z)|$ on C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside C.

It is noted that if all the zeros of an polynomial $P(z)=a_nz^n+a_{n-1}z^{n-1}+\cdots+a_0$ are inside the unit circle $|z|=1$, then all the zeros of the polynomial $z^nP(z^{-1})=a_0z^n+a_1z^{n-1}+\cdots a_n$ are outside the unit circle $|z|=1$.

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2. Polynomials with Positive and Negative Coefficients

Consider a polynomial

$$P(z)=a_n z^n+a_{n-1}z^{n-1}+\cdots+a_0,\ a_n\neq 0. \tag{1}$$

Theorem 1: (i) If $|a_n|>\sum_{i=0}^{n-1}|a_i|$, then all the zeros of $P(z)$ are inside the unit circle. (ii) If $|a_j| > \sum_{i=0,i\neq j}^{n}|a_i|$, where $j\neq n$, then at least one zero of $P(z)$ is not inside the unit circle.

Proof: (i) Lef $f(z)=a_n z^n$ and $g(z)=a_{n-1}z^{n-1}+\cdots+a_0$. Then on the unit circle ($z=e^{j\theta}$),

$$|g(z)|\leq|a_{n-1}|\,|z|^{n-1}+\cdots+|a_0|=\sum_{i=0}^{n-1}|a_i|,$$

and

$$|f(z)|=|a_n|.$$

It follows that $|f(z)|>|g(z)|$ on the unit circle. Since $f(z)$ has n zeros inside the unit circle, the same is true for $f(z)+g(z)=P(z)$ by Rouché's theorem.

(ii) Let $f(z)=a_j z^j$, where $j\neq n$, and $g(z)=\sum_{i=0,i\neq j}^{n} a_i z^i$. Using the argument similar to that given in the proof of (i) we have

$$|f(z)|>|g(z)|$$

on the unit circle. Since $f(z)$ has j zeros inside the unit circle and $j<n$, $P(z)=f(z)+g(z)$ also has j zeros inside the unit circle. Therefore, at least one zero of $P(z)$ is on or outside the unit circle. □

The coefficient requirements given in Theorem 1 are relaxed in Theorem 2. However, the test conditions are more complicated.

Theorem 2: Consder (1). If there exists a real number ρ, $|\rho|<1$, such that

(i) $|a_n|>\sum_{i=0}^{n}|a_{i-1}-\rho a_i|$, where $a_{-1}=0$, then all the zeros of $P(z)$ are inside the unit circle, and if

(ii) $|a_{j-1}-\rho a_j|>\sum_{i=0,i\neq j}^{n+1}|a_{i-1}-\rho a_i|$, where $j\neq n$, $a_{-1}=a_{n+1}=0$, then at least one zero of $P(z)$ is not inside the unit circle.

Proof: Consider the polynomial where $|\rho|<1$,

$$(z-\rho)P(z)=a_n z^{n+1}+(a_{n-1}-\rho a_n)z^n+\cdots+(a_0-\rho a_1)z-\rho a_0.$$

Therefore, the results follows from the conclusions of Theorem 1 and the fact that $|\rho|<1$. □

Remark: If $\rho=0$ in Theorem 2, then Theorem 2 becomes Theorem 1. If $\rho\neq 0$, some polynomials that do not satisfy Theorem 1 may satisfy Theorem 2.

Examples: Consider a polynomial

$$P(z)=z^3+0.2z^2-0.5z-0.38.$$

The polynomial $P(z)$ does not satisfy the conclusions of theorem 1. However, by choosing $\rho=0.1$, (i) of Theorem 2 is satisfied, i.e.

$$a_n=1>|0.038|+|-0.33|+|-0.52|+|0.1| =0.988.$$

On the other hand, the following polynomial

$$P(z)=z^3+0.8z^2-0.6z-0.6$$

does not satisfy the conclusions of Theorem 1, but with $\rho=0.5$ satisfies (ii) of Theorem 2 with $j=2$, since

$$\begin{aligned}|a_1-\rho a_2|&=|0.6+0.5(0.8)|\\&=1.0>|0.5(0.6)|+|-0.6+0.5(0.6)|\\&\quad+|0.8-0.5(1)|=0.9.\end{aligned}$$

The next theorem which follows from Theorem 1 has been considered by several authors (Heiner, 1985).

Theorem 3: Let

$$P(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1 z+a_0. \tag{2}$$

If there exists a real positive number ρ such that

$$\sum_{j=1}^{m}|\rho^{-j}a_{n-j}|<1,$$

then all the zeros of $P(z)$ are inside a circle of radius ρ, $|z|<\rho$.

Proof: Set $z=\rho w$ in $P(z)$ and define

$$Q(w)=\rho^{-n}P(\rho w)=w^n+\rho^{-1}a_{n-1}w^{n-1}+\cdots\rho^{-n+1}a_1 w+\rho^{-n}a_0.$$

Since, by hypothesis, $Q(w)$ satisfies the conclusion of Theorem 1, all the zeros of $Q(w)$ are inside the unit circle $|w|=1$. Since $|z|<\rho|w|$, the conclusion follows. □

The following simple coefficient property is a necessary condition of polynomials with all zeros in the unit circle.

Theorem 4: Consider (2). If all the zeros of $P(z)$ are inside the unit circle, then

$$|a_{n-j}|<\binom{n}{j},\ j=0,\ 1,\ \cdots,\ n-1.$$

Proof: The coefficient a_{n-j} of $P(z)$ is equal to the sum of all products of the n zeros of $P(z)$ taken j at a time. Since each zero is inside the unit circle, each product in a_{n-j} has a magnitude less than 1. Since there are $\binom{n}{j}$ such products in the expression for a_{n-j}, the conclusion follows. □

3. Polynomials with Positive Coefficients

In this section, we extend and give a simple proof to the well-known monotone coefficient theorem (Jury, 1964; Chottera and Jullien, 1982) of polynomials.

Theorem 5: Consider (1). If $a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0$, $a_{i+1} > a_i > a_{i-1}$ for at least one i, and $a_i>0$ for all i or $a_i<0$ for all i, then all the zeros of $P(z)$ are inside the unit circle.

Remark 1: The polynomial $P(z)$ cannot have a real positive zero, since if z_0 is real and positive, $P(z_0)>0$ if $a_0>0$ and $P(z_0)<0$ if $a_n<0$.

Remark 2: If $a_n=a_{n-1}=\cdots=a_0$, then $P(z)=a_n(z^n+z^{n-1}+\cdots+1)$. Since $(z-1)P(z)=a_n(z^{n+1}-1)$, all the zeros of $(z-1)P(z)$ are on the unit circle. Therefore, all the zeros of $P(z)$ are also on the unit circle.

Proof of Theorem 5 :Consider the polynomial

$$(z-1)P(z)=a_n z^{n+1}-[(a_n-a_{n-1})z^n + (a_{n-1}-a_{n-2})z_{n-1} + \cdots + (a_1-a_0)z+a_0],$$

where $a_i>0$ for all i (if $a_i<0$ for all i, consider $(z-1)(-P(z))$ in what follows). Therefore,

$$|(z-1)P(z)| \geq a_n|z|^{n+1}-|(a_n-a_{n-1})z^n + \cdots + (a_1-a_0)z+a_0|.$$

Since, by hypothesis, $a_{i+1}-a_i>0$ and $a_i-a_{i-1}>0$ for some i we have

$$|(a_{i+1}-a_i)z^i+(a_i-a_{i-1})z^{i-1}| < (a_{i+1}-a_i)|z|^i+(a_i-a_{i-1})|z|^{i-1}$$

for all z, where z is not real and positive. Therefore, increasing the value of this second absolute value term on the right in the expression for $|(z-1)P(z)|$ provides

$$|(z-1)P(z)| > a_n|z|^{n+1}-[(a_n-a_{n-1})|z|^n + \cdots + (a_1-a_0)|z|+a_0],$$

for all z, where z is not real and positive.

Increasing the bracket term on the right side using $|z|^j \geq |z|^{j-1}$, $j=1, 2, \cdots, n+1$ for $|z| \geq 1$, we have

$$|(z-1)P(z)| > a_n|z|^{n+1}-[(a_n-a_{n-1})|z|^{n+1} + \cdots + (a_1-a_0)|z|^{n+1}+a_0] = 0,$$

for all $|z| \geq 1$ and z not real and positive. Therefore, $P(z) \neq 0$ for $|z| \geq 1$ and for all z not real and positive. Since $P(z)$ has n zeros and no zero of $P(z)$ is real and positive, the zeros of $P(z)$ are located inside the unit circle. □

Corollary 1: If $a_0 \geq a_1 \geq \cdots \geq a_{n-1} \geq a_n$, $a_{i-1} > a_i > a_{i+1}$ for at least one i and $a_i>0$ for all i or $a_i<0$ for all i, then all the zeros of $P(z)$ are outside the unit circle

Proof: The conclusion follows using the argument given in the proof of the theorem on $z^nP(z-1)$ where $z \rightarrow z^{-1}$. □

Corollary 2: If the coefficients of $P(z)$ satisfy $a_{n-j}>0$ for $j=0, 2, \cdots$, and $a_{n-j}<0$ for $j=1, 3, \cdots, n$,

where

$$a_n \geq -a_{n-1} \geq \cdots \geq a_0,$$

and

$$|a_{i+1}|>|a_i|>|a_{i-1}|,$$

for at least one i, then all the zeros of $P(z)$ are inside the unit circle.

Proof: Since the polynomial $P(-z)$ for n even or $-P(-z)$ for n odd satisfy the hypothesis of the theorem, the conclusion follows.

The final theorem gives a simple test on the coefficients of a polynomial which, if satisfied, guarantees the zeros of the polynomial lie inside the unit circle. □

Theorem 6: If all the coefficients of (1) are positive and if

$$\frac{a_{n-1}+a_{n-2}+\cdots+a_0-a_n}{a_n+a_{n-1}+\cdots+a_1-a_0} < \min_{0 \leq j \leq n-1} \frac{a_{n-j-1}}{a_{n-j}},$$

then all the zeros of $P(z)$ are inside the unit circle.

Proof: Consider the polynomial

$$(z-\rho)P(z)=a_n z^{n+1}+(a_{n-1}-\rho a_n)z^n + \cdots + (a_0-\rho a_1)z-\rho a_0.$$

Set

$$\rho=\min_{0\leq j\leq n-1}\frac{a_{n-j-1}}{a_{n-j}},$$

then

$$\frac{a_{n-1}+a_{n-2}+\cdots+a_0-a_n}{a_n+a_{n-1}+\cdots+a_1-a_0}<\rho.$$

Cross-multiplying and combining terms results in

$$-a_n<-(a_{n-1}-\rho a_n)-\cdots-\cdots(a_0-\rho a_1)-\rho a_0.$$

Therefore,

$$a_n>\sum_{i=1}^{n}(a_{i-1}-\rho a_i)+\rho a_0.$$

Since, by the hypothesis,

$$a_{i-1}-\rho a_i=a_i(\frac{a_{i-1}}{a_i}-\rho)\geq 0,$$

for $i=1, 2, \cdots, n$, the coefficients satisfy Theorem 2. Therefore all the zeros of $P(z)$ are inside the unit circle.

Example: The following polynomial does not satisfy Theorem 1.

$$P(z)=z^3+0.4z^2+0.5z+0.2$$

Since the coefficients of $P(z)$ are not monotone, Theorem 5 is not applicable. Therefore, we consider the conditions of Theorem 6. We determine

$$\rho=\min\{\frac{a_2}{a_3}, \frac{a_1}{a_2}, \frac{a_0}{a_1}\}=\min\{0.4, 1.25, 0.4\}$$
$$=0.4,$$
$$\frac{a_2+a_1+a_0-a_3}{a_3+a_2+a_1-a_0}=\frac{0.4+0.5+0.2-1}{1+0.4+0.5-0.2}$$
$$=\frac{0.1}{1.7}=0.058.$$

Therefore, all the zeros of $P(z)$ are inside the unit circle since the condition of Theorem 6 is satisfied.

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